An extension of the Lyapunov analysis for the predictability problem

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The predictability problem for systems with different characteristic time scales is investigated. It is shown that even in simple chaotic dynamical systems, the leading Lyapunov exponent is not sufficient to estimate the predictability time. This fact is due the saturation of the error on the fast components of the system which therefore do not contribute to the exponential growth of the error at large errors. It is proposed to adopt a generalization of the Lyapunov exponent which is based on the natural concept of error growing time at finite error size. The method is first illustrated on a simple numerical model obtained by coupling two Lorenz systems with different time scales. As a more realistic example, this analysis is then applied to a toy model of Atmospheric circulation recently introduced by Lorenz.

I. INTRODUCTION

The prediction of the future state of a system known the actual state is a fundamental problem with obvious applications in geophysical flows (Leith, 1975; Leith and Kraichnan, 1972; Leith, 1978). There are many limitations to the ability of predicting the state of a geophysical system, e.g. the atmosphere, one of the most important is the lack of knowledge, or the difficulty of full implementation, of the equations of motion. Still, even if one assume to perfectly know the system and to have sufficiently large computers, the predictability can be severely limited by the dynamics itself, i.e. the "intrinsic unpredictability" the a system which is the subject of our study.

A well known, and very popular, example of low-predictable system is given by a chaotic system (Lorenz, 1963). By definition, chaotic dynamical systems display sensible dependence on initial conditions: two initially close trajectories will diverge exponentially in the phase space with a rate given by the leading Lyapunov exponent λ_{max} (see Eckmann and Ruelle, 1985). Because the initial condition can be measured only with a finite uncertainty δ, we can know the future state of the system at a tolerance level Δ only up to a maximum time

$$T_p \sim \frac{1}{\lambda_{max}} \ln\left(\frac{\Delta}{\delta}\right) \tag{1}$$

One important consequence of equation (1) is that the predictability time has a very weak dependence on the precision of the initial condition and on the tolerance, therefore the predictability time is an intrinsic quantity of the system as the Lyapunov exponent is.

The naive formula (1) for the predictability problem holds only for infinitesimal perturbations and in non intermittent systems; in the general case one has a series of problems and subtle points which have been objected of several studies in last years (Crisanti et al., 1993; Aurell et al., 1996, 1997). One delicate issue is particularly relevant for our present study and essentially says that, although the Lyapunov exponent for the atmosphere (as a whole) is presumably rather large (due to the small scale turbulence), the large scale behavior of the system can be forecasted with good accuracy for several days (Lorenz, 1969; Lorenz, 1982; Simmons et al., 1995).

The apparent paradox comes from the identification of the predictability time with the inverse of the Lyapunov exponent based on equation (1) which is actually of little relevance even in few degree-of-freedom dynamical systems. Indeed, in presence of different characteristic time scales, as is the case in any realistic model of geophysical flows, the Lyapunov exponent will be roughly proportional to the inverse smallest characteristic time. This time is associated to the smallest, low energy containing scales which, after the fast saturation, do not play a role any more in the error growth law. Large errors will grow, in general, with the characteristic time of the largest, energy containing scales (Leith, 1971; Leith and Kraichnan, 1972). Thus when the initial error is not very small, as is often the case in a predictability experiment, the leading Lyapunov exponent may play no role at all.

To be more quantitative, in this paper we investigate the predictability problem in two time scale dynamical systems. We apply a recently introduced generalization of the Lyapunov exponent to finite perturbations. We will show that

the Finite Size Lyapunov Exponent (FSLE) is more suitable for characterizing the predictability of complex systems where the growth rate of large errors in not ruled by the Lyapunov exponent.

The models considered here are crude approximations of a realistic geophysical flow also because both the subsystems have a single time scale. It would be interesting to extend the investigation to more realistic situations and comparing the latter case with present results.

This remaining of the paper is organized as follows: in section II we introduce the Finite Size Lyapunov Exponent which is applied to the system models in section III. Section IV is devoted to conclusions.

II. THE FINITE SIZE LYAPUNOV EXPONENT

The notion of Lyapunov exponent is based on the average rate of exponential separation of two infinitesimally close trajectory in the phase space:

$$\lambda_{max} = \lim_{t\to\infty} \lim_{\delta x(0)\to 0} \frac{1}{t} \ln \frac{\delta x(t)}{\delta x(0)} \tag{2}$$

where $\delta x(t)$ is the distance between the trajectories with a suitable norm and the two limits cannot be interchanged. The standard algorithm (Benettin et al., 1980) for computing the Lyapunov exponent is based on (2), with the trick of periodical rescaling of the two trajectory in order to keep their distance "infinitesimal".

As already discussed in the previous section, the second limit in (2) is of dubious interest in the predictability problem because the initial incertitude on the system variables is in general not infinitesimal. Therefore one would like to relax the infinitesimal constrain still maintaining some well defined mathematical properties. Recently, a generalization of (2) which allows to compute the average exponential separation of two trajectories at finite errors δ have been introduced. The Finite Size Lyapunov Exponent, $\lambda(\delta)$, is based on the concept of error growing time $T_r(\delta)$ which is the time it takes for a perturbation of initial size δ to grow of a factor r. The ratio r should not be taken too large, in order to avoid the growth through different scales. The error growing time is a fluctuating quantity and one has to take the average along the trajectory as in (2). The Finite Size Lyapunov Exponent is then defined as

$$\lambda(\delta) = \left\langle \frac{1}{T_r(\delta)} \right\rangle_t \ln r = \frac{1}{\langle T_r(\delta)\rangle} \ln r \tag{3}$$

where $\langle ... \rangle_t$ denotes the natural measure along the trajectory and $\langle ... \rangle$ is the average over many realizations. For an exhaustive discussion on the way to take averages, see Aurell et al. (1997).

In the limit of infinitesimal perturbations, $\delta \to 0$, definition (3) reduces to that of the leading Lyapunov exponent (2). In practice, $\lambda(\delta)$ displays a plateau at the value λ_{max} for sufficiently small δ.

To practically compute the FSLE, one has first to define a series of thresholds $\delta_n = r^n \delta_0$, and to measure the time $T_r(\delta_n)$ that a perturbation with size δ_n takes to grow up to δ_{n+1}. The time $T_r(\delta_n)$ is obtained by following the evolution of the perturbation from its initial size $\delta_{\min}$ up to the largest threshold $\delta_{\max}$. This is done by integrating two trajectories of the system that start at an initial distance $\delta_{\min}$. In general, one must take $\delta_{\min} \ll \delta_0$, in order to allow the direction of the initial perturbation to align with the most unstable phase-space direction. The FSLE, $\lambda(\delta_n)$, is then computed by averaging the error growing times over several realizations according to (3).

Note that the FSLE has conceptual similarities with the ϵ-entropy (Kolmogorov, 1956; see also Gaspard and Wang, 1993). This latter measures the bandwidth that is necessary for reproducing the trajectory of a system within a finite accuracy δ. The ϵ-entropy approach has already been applied to the analysis of simple systems and experimental data (Gaspard and Wang, 1993), giving interesting results. The calculation of the ϵ-entropy is, however, much more expensive from a computational point of view and of little relevance for the predictability problem.

The computation of the FSLE gives information on the typical predictability time for a trajectory with initial incertitude δ. To be more quantitative, one can introduce the average predictability time from an initial error δ to a given tolerance Δ as the average error growing time, i.e.

$$T_p = \int_{\delta}^{\Delta} \frac{d \ln \delta'}{\lambda(\delta')} \tag{4}$$

which reduces to (1) in the case of constant λ. From general considerations, one expects that $\lambda(\delta)$ is a decreasing function of δ and thus (4) gives longer predictability time than (1).

III. THE MODELS

We now discuss the application of the FSLE analysis to two relatively simple dynamical systems presenting different characteristic time scales. The proposed models are of little physical relevance; they should rather be intended as prototypical models for the predictability problem in complex flows.

The first example is obtained by coupling two Lorenz models (Lorenz, 1963), the first representing the slow dynamics and the second the fast dynamics

$$\begin{cases} \frac{dx_1^{(s)}}{dt} &= \sigma(x_2^{(s)} - x_1^{(s)}) \\ \frac{dx_2^{(s)}}{dt} &= (-x_1^{(s)}x_3^{(s)} + r_s x_1^{(s)} - x_2^{(s)}) - \epsilon_s x_1^{(f)} x_2^{(f)} \\ \frac{dx_3^{(s)}}{dt} &= x_1^{(s)}x_2^{(s)} - bx_3^{(s)} \\ \frac{dx_1^{(f)}}{dt} &= c\sigma(x_2^{(f)} - x_1^{(f)}) \\ \frac{dx_2^{(f)}}{dt} &= c(-x_1^{(f)}x_3^{(f)} + r_f x_1^{(f)} - x_2^{(f)}) + \epsilon_f x_1^{(f)} x_2^{(s)} \\ \frac{dx_3^{(f)}}{dt} &= c(x_1^{(f)}x_2^{(f)} - bx_3^{(f)}) \end{cases} \tag{5}$$

The choice of the form of the coupling is constrained by the physical request that the solution remains in a bounded region of the phase space. Since

$$\frac{d}{dt}\left\{\epsilon_f\left(\frac{(x_1^{(f)})^2}{2\sigma} + \frac{(x_2^{(f)})^2}{2} + \frac{(x_3^{(f)})^2}{2} - (r_f+1)x_3^{(f)}\right) + \right.$$
$$\left.\epsilon_s\left(\frac{(x_1^{(s)})^2}{2\sigma} + \frac{(x_2^{(s)})^2}{2} + \frac{(x_3^{(s)})^2}{2} - (r_s+1)x_3^{(s)}\right)\right\} < 0, \tag{6}$$

if the trajectory is enough far from the origin, one has that it evolves in a bounded region of the phase space. The parameters have the values $\sigma = 10$, $b = 8/3$ and $c = 10$, the latter giving the relative time scale between the fast and slow dynamics. The two Rayleigh numbers are taken different, $r_s = 28$ and $r_f = 45$, in order to avoid sincronization effects.

With the present choice, the two uncoupled systems ($\epsilon_s = \epsilon_f = 0$) display chaotic dynamics with Lyapunov exponent $\lambda^{(f)} \simeq 12.17$ and $\lambda^{(s)} \simeq 0.905$ respectively and thus a relative intrinsic time scale of order 10.

By switching on the couplings ϵ_s and ϵ_f we obtain a single dynamical system whose maximal Lyapunov exponent λ_{max} is close (for small couplings) to the Lyapunov exponent of the faster decoupled system ($\lambda^{(f)}$). We will consider a single realization of the couplings, with $\epsilon_f = 10$ and $\epsilon_s = 10^{-2}$. The global Lyapunov exponent is found to be in this case $\lambda_{max} \simeq 11.5$ which is indeed close to $\lambda^{(f)}$ in the uncoupled case. With the present choice of the couplings, the fast dynamics is driven by means of the effective Rayleigh number $r_{eff} = r_f + \epsilon_f x_2^{(s)}(t)/c$ and one recognize in the time evolution the slow-varying component of the driver (see figure 1).

For what concern the predictability, one expect reasonably that for small coupling ϵ_s the slow component of the system $\boldsymbol{x}_s$ remains predictable up to its own characteristic time. On the other hand, for any coupling $\epsilon \neq 0$ we obtain a single dynamical system in which the errors grow with the leading Lyapunov exponent $\lambda_{max} \simeq \lambda^{(f)}$. The apparent paradox stems from saturation effects which becomes apparent as soon as one is interested in non infinitesimal errors.

We have integrated two trajectories of (5) starting from very close initial conditions. One trajectory represents the "true" (reference) trajectory $\boldsymbol{x}$ and the other is the forecast (perturbed trajectory $\boldsymbol{x}'$) subjected to an initial error $\delta x(0)$. The error is computed here by means of the Euclidean distance in the phase space

$$\delta x(t) = \left(\delta x_f(t)^2 + \delta x_s(t)^2\right)^{1/2} = \left[\sum_{i=1}^{3}\left(x_i'^{(f)} - x_i^{(f)}\right)^2 + \sum_{i=1}^{3}\left(x_i'^{(s)} - x_i^{(s)}\right)^2\right]^{1/2} \tag{7}$$

Figure 2 reports the results for the error growth averaged over 500 experiments with $\delta x_f(0) = 10^{-8}$ and $\delta x_s(0) = 10^{-12}$. We observe that the relative magnitude of the initial errors is irrelevant for what concerns small errors because the error direction in the phase space will be rapidly aligned toward the most unstable direction. For small times ($t \leq 2$), both the errors can be considered infinitesimal and the growth rate is thus given by the global Lyapunov exponent λ_{max}. This is the linear regime of the error growth in which the Lyapunov exponent is the relevant parameter

for the predictability. For larger times, the fast component of the error, δx_f, reaches the saturation, the trajectories separation evolves according to the full non linear equations of motion and the growth rate for the slow component is strongly reduced. From figure 2 one observes that the slow component error δx_s is still well below the saturation value, and grows with a rate close to its characteristic inverse time $\lambda^{(s)}$.

We now apply the FSLE algorithm to the slow component of the the error, δx_s (figure 3). We define a series of $m = 25$ thresholds starting with $\delta_0 = 10^{-6}$ and ratio $r = 2$. The results presented (figure 3) are obtained after averaging over $N = 500$ realizations. For very small δ, the FSLE recovers the leading Lyapunov exponent λ_{max}, indicating that in small scale predictability the fast component has indeed a dominant role. As soon as the error grows above the coupling ϵ_s, $\lambda(\delta)$ drops to a value close to $\lambda^{(s)}$ and the characteristic time of small scale dynamics is no more relevant.

In figure 4 we plot the slow component predictability time (4) for a fixed initial error $\delta x_s = 10^{-6}$ as a function of the tolerance Δ. We observe, as expected, an enhancement of T_p as soon as one accepts a tolerance larger than the typical fast component fluctuation in the slow time series. Observe that the naive application of (1) would heavily underestimate the predictability time for large tolerance (dashed line).

We now consider the second example. It is a more complex system introduced by Lorenz (Lorenz, 1996) as a toy model for the Atmosphere dynamics which includes explicitly both large scales (synoptic scales, slow component) and small scales (convective scales, fast component). The apparent paradox described above can be reformulated here by saying that a more refined Atmosphere model (which is able to capture the small scale dynamics) would be less predictable of a rougher one (which resolve only large scale motion) and thus the latter should be preferred for numerical weather forecasting. We will see that also in this case, the effect of the small, fast evolving, scales becomes irrelevant for the predictability of large scale motion if one consider large errors.

The model introduces a set of large scale, slow evolving, variables x_k and small scale, fast evolving, variables $y_{j,k}$ with $k = 1, ..., K$ and $j = 1, ..., J$. As in (Lorenz, 1996) we assume periodic boundary conditions on k ($x_{K+k} = x_k$, $y_{j,K+k} = y_{j,k}$) while for j we impose $y_{J+j,k} = y_{j,k+1}$. The equation of motion are

$$\begin{aligned} \frac{dx_k}{dt} &= -x_{k-1}\left(x_{k-2} - x_{k+1}\right) - x_k + F - \textstyle\sum_{j=1}^{J} y_{j,k} \\ \frac{dy_{j,k}}{dt} &= -cby_{j+1,k}\left(y_{j+2,k} - y_{j-1,k}\right) - cy_{j,k} + x_k \end{aligned} \tag{8}$$

in which c again represent the relative time scale between fast and slow dynamics and b is a parameter which controls the relative amplitude.

Let us note that (8) has the same qualitative structure of a finite mode truncation of Navier-Stokes equation, with quadratic inertial terms and viscous dissipation. The coupling (with unit strength) is chosen in order to have the "energy"

$$E = \frac{1}{2}\left(\sum_{k=1}^{K} x_k^2 + \sum_{k=1}^{K}\sum_{j=1}^{J} y_{j,k}^2\right) \tag{9}$$

conserved in the inviscid, unforced limit. The forcing term drives only the large scales and we will consider $F = 10$ which is sufficient for developing chaos.

We have performed the computation of the FSLE for system (8) with parameters as in (Lorenz, 1996): $K = 36$, $J = 10$, $c = b = 10$ implying that the typical y variable is 10 times faster and smaller than the x variable. In this case we choose to adopt for measuring the errors the global Euclidean norm on both the slow and fast variables (energy norm): this is for mimic a realistic situation in which we are not able to recognize *a priori* the slow component in the system.

The result of the FSLE computation is displayed in figure 5 after averaging over $N = 1000$ realizations with initial error $\delta_{min} = 10^{-5}$. We set $m = 20$ thresholds with $\delta_0 = 10^{-3}$ and ratio $r = 2^{1/2}$. For very small errors we observe the saturation of $\lambda(\delta)$ to the leading Lyapunov exponent of the system $\lambda_{max} \simeq 9.9$. For errors larger than the typical r.m.s. value of the fast variables ($\langle y^2 \rangle^{1/2} \simeq 0.25$) we observe a second plateau at $\lambda \simeq 0.5$, corresponding to the inverse characteristic time of large scales. We observe that the relative time scale between fast and slow motions as computed by the FSLE is slightly larger than the value of the parameter c. We think that this effect is due to coupling which here cannot be assumed small as in the previous example.

In figure 6 we plot the predictability time (4) for fixed initial error $\delta = 10^{-3}$ and different thresholds. As in the previous example, we observe an enhancement of the predictability time for large tolerance Δ with respect to the Lyapunov exponent estimation. For large initial errors (as it is usually the case in numerical weather forecasting) the predictability time is thus independent of the Lyapunov exponent.

IV. CONCLUSIONS

We have shown that in systems with possess different characteristic time scales, the predictability time can be an independent quantity of the leading Lyapunov exponent. The latter is usually associated to the faster characteristic time and dominates the exponential growth of infinitesimal errors. Large errors will evolve in general with large scale characteristic time which thus rules large scale predictability.

We have introduced a generalization of the Lyapunov exponent which allows to compute the average exponential error growth at a given error size δ. The Finite Size Lyapunov Exponent is expected to converge at the leading Lyapunov exponent for very small errors. For larger errors, $\lambda(\delta)$ is decreasing with δ and thus the FSLE analysis predicts an enhancement of the predictability time as observed in several numerical experiments.

We illustrate these concepts on two model examples which possess different characteristic timescales. The numerical computation of the FSLE confirms the predictability enhancement with respect to the Lyapunov analysis.

Our results have a general significance which exceeds the proposed models. In particular, whenever one can identify in the system different features with different intrinsic time scales, one expects that slow varying quantities (i.e. large scale features) are predictable longer than fast evolving quantities. Moreover, our results demonstrate that the estimation of the predictability time for a large scale circulation model do not require to resolve the small scale dynamics.

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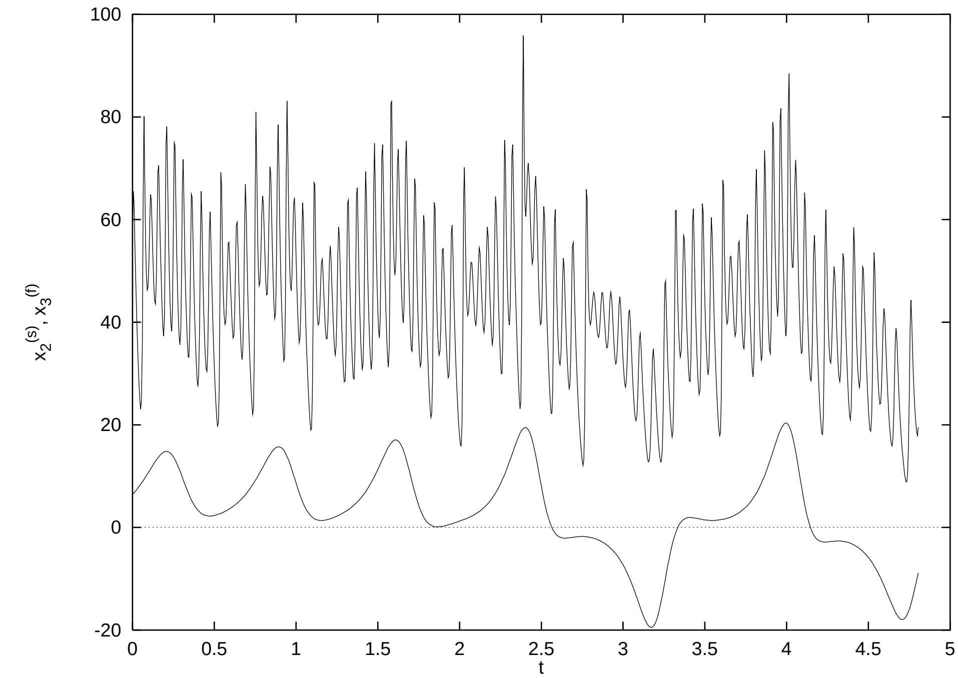

FIG. 1. Time series of the slow variable y_s (lower curve) and of the fast variable z_f (upper curve) on the attractor.

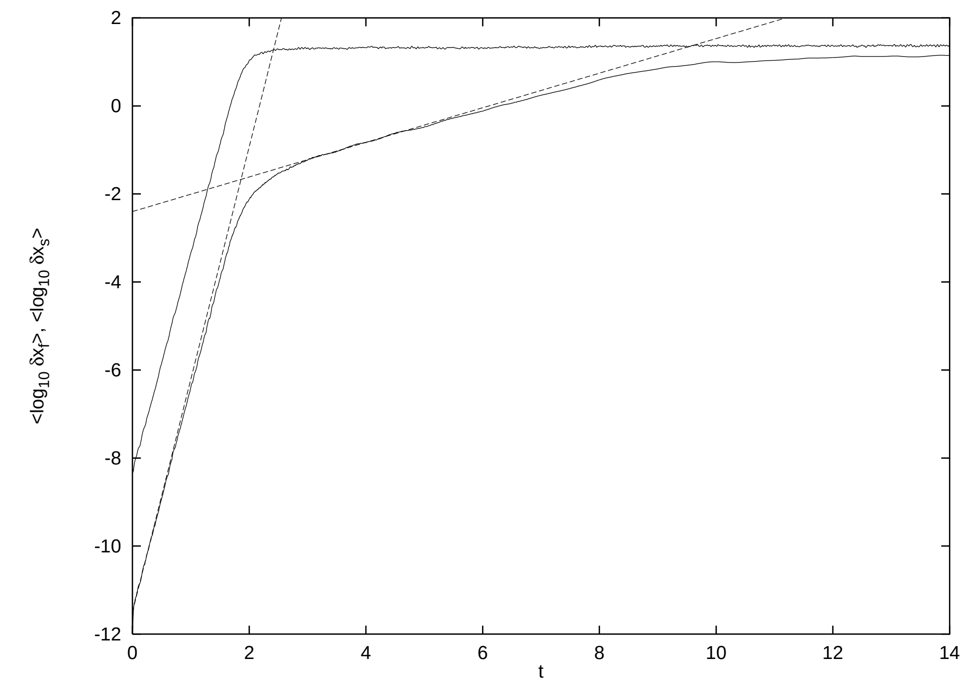

FIG. 2. Typical error growth for the fast component δx_f (upper curve) and for the slow component δx_s in the coupled Lorenz models with $\delta x_f(0) = 10^{-8}$ and $\delta x_s(0) = 10^{-12}$, averaged over 500 samples. In order to detect the typical behavior we compute the average of the logarithm. The dashed lines show the exponential growths with exponents $\lambda^{(f)}$ and $\lambda^{(s)}$.

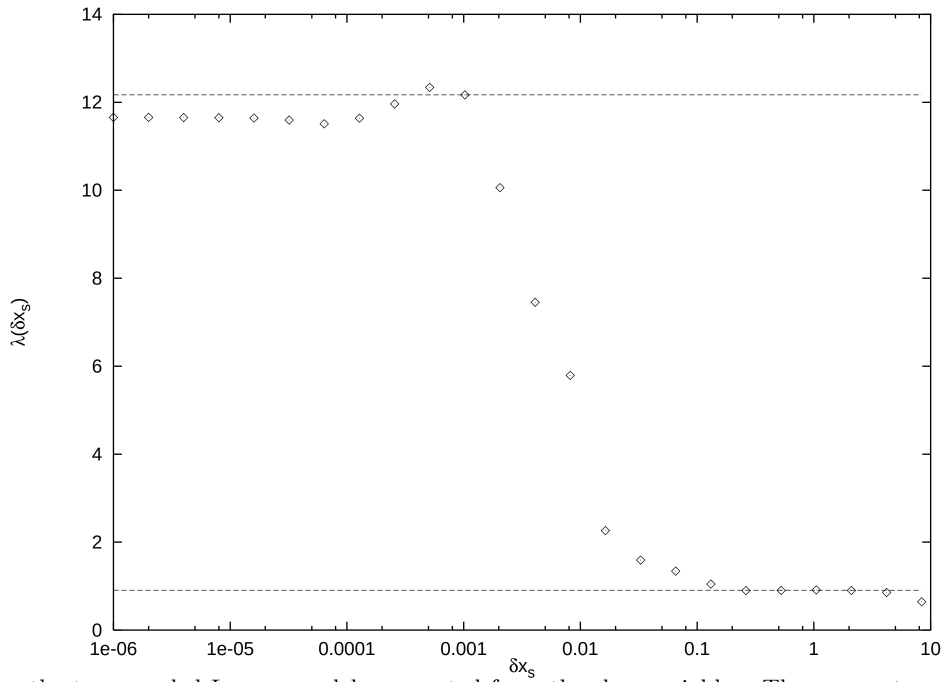

FIG. 3. FSLE for the two coupled Lorenz models computed from the slow variables. The parameters for the computation are: $\delta_0 = 10^{-6}$, $m = 25$, $r = 2$ and $N = 500$. The two horizontal lines represent the uncoupled Lyapunov exponents $\lambda^{(f)}$ and $\lambda^{(s)}$.

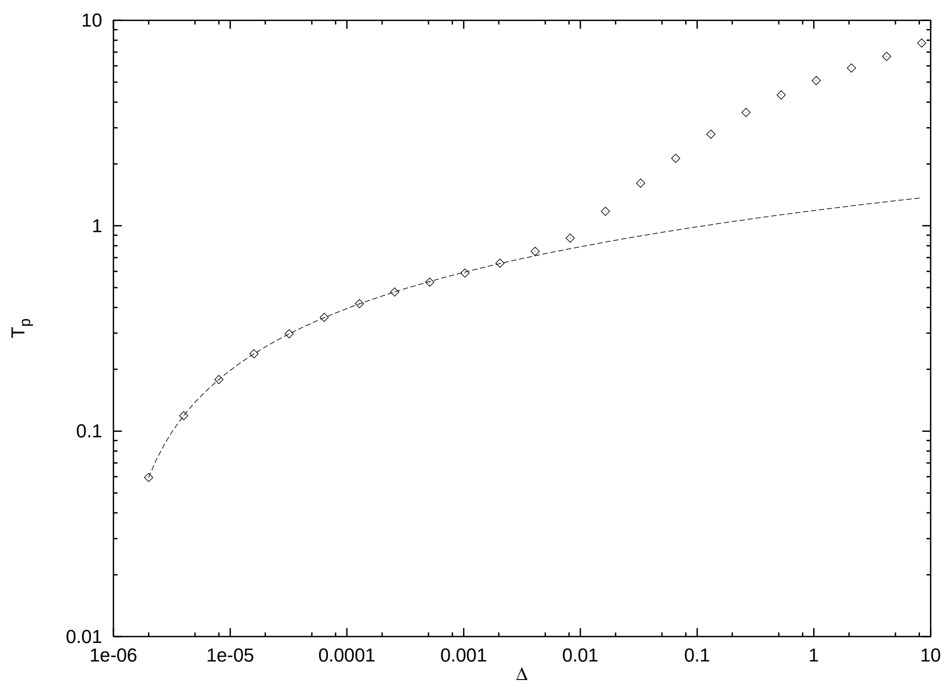

FIG. 4. Predictability time for the slow component of the two coupled Lorenz models as a function of the tolerance Δ. The initial error is fixed at $\delta = 10^{-6}$. The dashed line represent the Lyapunov estimation $T_p \sim \lambda^{-1} \ln(\Delta/\delta)$.

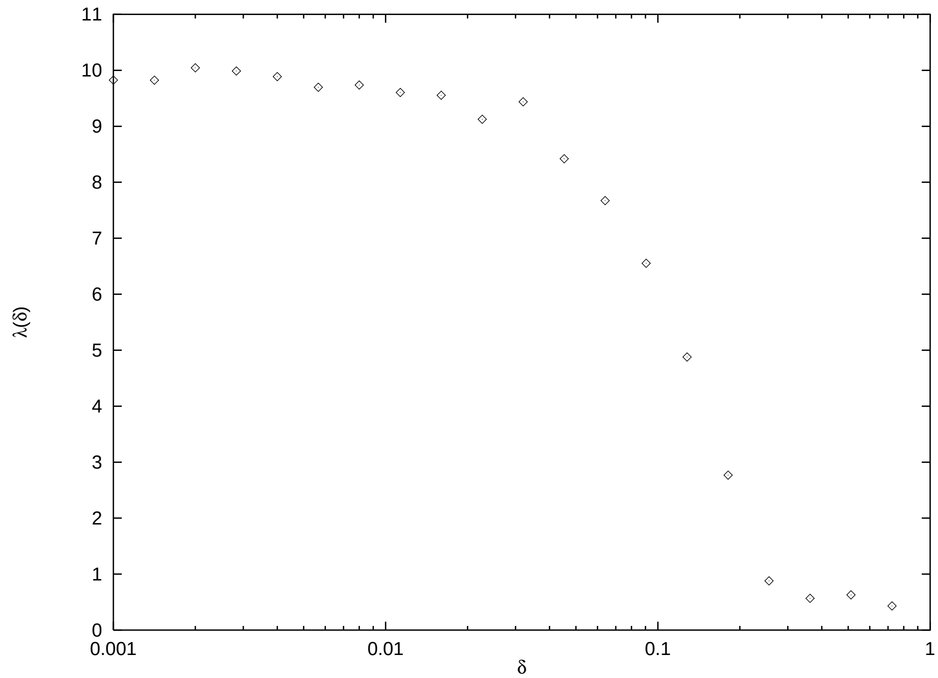

FIG. 5. FSLE computed for the toy Atmospheric model. The parameters for the computation are: $\delta_0 = 10^{-3}$, $m = 20$, $r = 2^{1/2}$ and $N = 1000$.

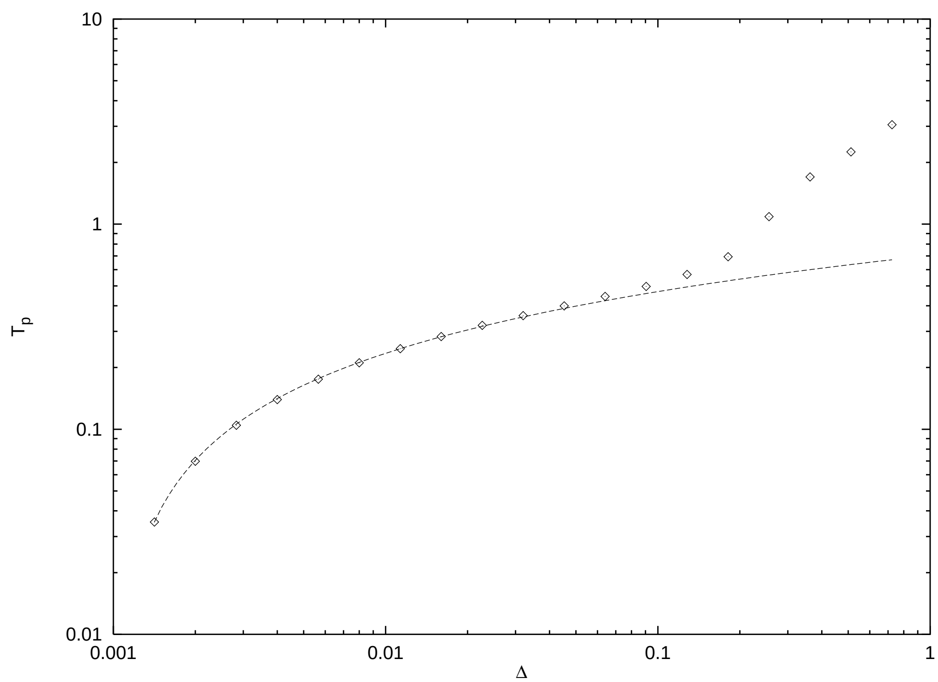

FIG. 6. Predictability time for the toy Atmospheric model as a function of the tolerance Δ. The initial error is $\delta = 10^{-3}$. The dashed line represent the Lyapunov based estimation.